

## Supplement to “Stability in repeated matching markets”

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### APPENDIX S.1: PROOF OF THEOREM 2

The proof of Theorem 2 follows a similar construction to that in [Fudenberg and Maskin \(1986\)](#) and [Abreu, Dutta, and Smith \(1994\)](#). Due to the notational complexity of the proof, I first illustrate the self-enforcing matching process using a phase diagram, before proceeding with the full proof of Theorem 2.

#### S.1.1 A phase diagram illustration

Consider a case when there are only two firms  $\mathcal{F} = \{f, f'\}$ . Figure S.1 is a phase diagram illustrating the self-enforcing matching process.

In this phase diagram,  $\lambda \in \Lambda^*$  is the random matching we want to sustain on the path of play. The matching  $\underline{m}_f \in \arg \min_{m \in M_{\mathcal{R}}^o} \max_{W \subseteq D_f(m), |W| \leq q_f} u_f(W)$  is the minmax matching for firm  $f$ , and  $\underline{m}_{f'}$  is defined similarly.

The random matchings  $\lambda_f$  and  $\lambda_{f'}$  are “firm-specific punishments” that are played after the minmax phase. In particular, they are random matchings that guarantee the properties

$$u_f(\lambda_f) < u_f(\lambda) \quad \text{and} \quad u_{f'}(\lambda_f) < u_{f'}(\lambda).$$

In other words, each firm prefers the on-path randomization  $\lambda$  over their own firm-specific punishments, and each firm prefers the other firm being punished over being punished itself.

The existence of  $\lambda_f$  and  $\lambda_{f'}$  can be shown by resorting to the non-equivalent utilities (NEU) condition in [Abreu, Dutta, and Smith \(1994\)](#): Observe that for each firm, when it is unmatched, it is indifferent toward how the other firm  $f'$  matches with workers, so their utilities cannot be positive affine transformation of another. Lemma 1 and Lemma 2 in [Abreu, Dutta, and Smith \(1994\)](#) then ensure the existence of  $\lambda_f$  and  $\lambda_{f'}$  that satisfy the properties above.

#### S.1.2 Complete proof of Theorem 2

Fix  $\lambda^0 \in \Lambda^*$ . Define  $u^0 = u(\lambda^0)$  and  $U^* \equiv \{u(\lambda) : \lambda \in \Lambda^*\}$ . Observe that for firms in  $\mathcal{F} \cap \mathcal{R}$ , the set  $M_{\mathcal{R}}^o$  satisfies the NEU condition in [Abreu, Dutta, and Smith \(1994\)](#): holding  $f \in$

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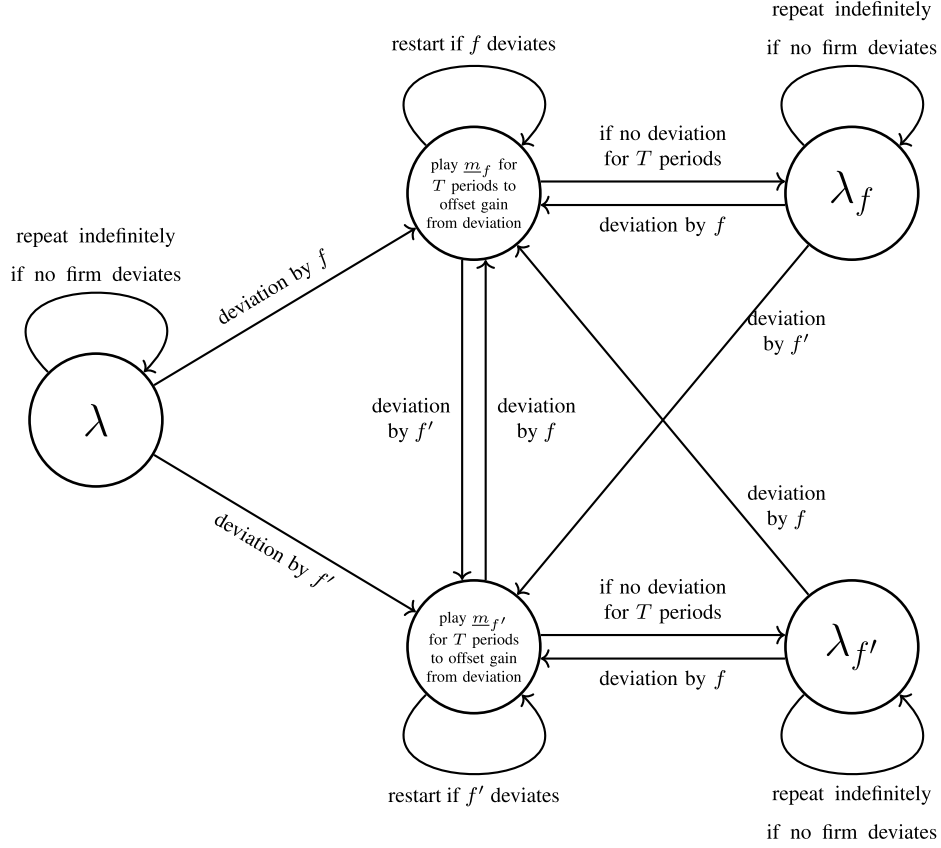


FIGURE S.1. A Phase Diagram Representation.

$\mathcal{F} \cap \mathcal{R}$  unmatched,  $f$  is indifferent toward how another firm  $f' \in \mathcal{F} \cap \mathcal{R}$  matches with workers, so their utilities cannot be positive affine transformation of another. Lemma 1 and Lemma 2 in [Abreu, Dutta, and Smith \(1994\)](#) then ensure the existence of vectors  $\{u^f : f \in \mathcal{F} \cap \mathcal{R}\} \subseteq U^*$ , such that

$$u_f^f < u_f^0 \quad \text{and} \quad u_f^f < u_f^{f'}$$

for all  $f, f' \in \mathcal{F} \cap \mathcal{R}$  and  $f \neq f'$ . Let  $\lambda^f \in \Lambda^*$  be the distribution over  $M_{\mathcal{R}}$  that gives rise to the payoff vector  $u^f$  for each  $f$ . In addition, for each  $f \in \mathcal{F} \cap \mathcal{R}$ , let

$$\underline{m}_f \in \arg \min_{m \in M_{\mathcal{R}}^0} \max_{W \subseteq D_f(m), |W| \leq q_f} u_f(W)$$

be the stage-game recommendation to minmax firm  $f$ .

Consider the matching process represented by the automaton  $(\Theta, \gamma^0, f, \gamma)$ , where the following statements hold:

- (i) The equality  $\Theta = \{\theta(e, m) : e \in \mathcal{F} \cap \mathcal{R} \cup \{0\}, m \in M_{\mathcal{R}}\} \cup \{\theta(f, t) : f \in \mathcal{F} \cap \mathcal{R}, 0 \leq t < L\}$  is the set of all states.

- (ii) The variable  $\gamma^0$  is the initial distribution over states, which satisfies  $\gamma^0(\theta(0, m)) = \lambda^0(m)$  for all  $m \in M_{\mathcal{R}}^{\circ}$ .
- (iii) The function  $O : \Theta \rightarrow M$  is the output function, where  $O(\theta(e, m)) = m$  and  $O(\underline{\theta}(f, t)) = \underline{m}_f$ .
- (iv) The function  $\kappa : \Theta \times M \rightarrow \Delta(\Theta)$  is the transition function. For states  $\{\underline{\theta}(f, t) \mid 0 \leq t < L - 1\}$ ,  $\kappa$  is defined as

$$\begin{aligned} & \kappa(\underline{\theta}(f, t), m') \\ &= \begin{cases} \underline{\theta}(f', 0) & \text{if } m' \neq \underline{m}_f; m' = [\underline{m}_{f'}, (f', W)] \text{ for some } f' \in \mathcal{F} \cap \mathcal{R} \text{ and } W \subseteq \mathcal{W} \\ \underline{\theta}(f, t + 1) & \text{otherwise.} \end{cases} \end{aligned}$$

For states  $\underline{\theta}(f, L - 1)$ , the transition is defined as

$$\begin{aligned} & \kappa(\underline{\theta}(f, L - 1), m') \\ &= \begin{cases} \underline{\theta}(f', 0) & \text{if } m' \neq \underline{m}_f; m' = [\underline{m}_{f'}, (f', W)] \text{ for some } f' \in \mathcal{F} \cap \mathcal{R} \text{ and } W \subseteq \mathcal{W} \\ \gamma^f & \text{otherwise,} \end{cases} \end{aligned}$$

where for each  $f \in \mathcal{F} \cap \mathcal{R}$ ,  $p^f$  is the distribution over states that satisfies  $\gamma^f(\theta(f, m)) = \lambda^f(m)$  for all  $m \in M$ .

For states  $\theta(e, m)$ , the transition is

$$\begin{aligned} & \kappa(\theta(e, m), m') \\ &= \begin{cases} \underline{\theta}(f', 0) & \text{if } m' \neq \underline{m}_f; m' = [\underline{m}_{f'}, (f', W)] \text{ for some } f' \in \mathcal{F} \cap \mathcal{R} \text{ and } W \subseteq \mathcal{W} \\ \gamma^e & \text{otherwise,} \end{cases} \end{aligned}$$

where the distributions  $\gamma^e$  are defined as above.

Note that owing to the identifiability of the deviating firm (Lemma 2), for any  $\theta \in \Theta$  and matching  $m' \neq O(\theta)$ , we can uniquely identify the firm responsible for this deviation, so the transition above is well defined.

Note that no firms in  $\mathcal{T}$  wish to deviate, since they are always matched with their top coalition workers; no workers want to deviate since all recommended matchings are in  $M_{\mathcal{R}}^{\circ}$ . It remains to verify that no firm  $f \in \mathcal{F} \cap \mathcal{R}$  has incentives to deviate. Choose a number  $Z > \sup_{\{m \in M, f \in \mathcal{F} \cap \mathcal{R}\}} u_f(m)$ .

*For states of the form  $\{\theta(e, m)\}$*  Consider a one-shot deviation  $(f, W)$ . There are two cases to consider.

*Case 1:  $f \neq e$ .* Without deviation,  $f$  has value  $(1 - \delta)u_f(m) + \delta u_f^e$ . After deviation,  $f$  yields less than  $(1 - \delta)Z + \delta(1 - \delta^L)\underline{u}_f^{\mathcal{R}} + \delta^{L+1}u_f^f$ . There is no profitable one-shot deviation for  $f$  if

$$(1 - \delta)u_f(m) + \delta u_f^e \geq (1 - \delta)Z + \delta(1 - \delta^L)\underline{u}_f^{\mathcal{R}} + \delta^{L+1}u_f^f.$$

As  $\delta \rightarrow 1$ , the LHS converges to  $u_f^e$  while the RHS converges to  $u_f^f$ . By construction,  $u_f^e > u_f^f$ , so such deviations are not profitable for  $\delta$  high enough.

*Case 2:  $f = e$ .* Without deviation,  $f$  has value  $(1 - \delta)u_f(m) + \delta u_f^f$ . After deviation,  $f$  yields less than  $(1 - \delta)Z + \delta(1 - \delta^L)\underline{u}_f^{\mathcal{R}} + \delta^{L+1}u_f^f$ . There is no profitable one-shot deviation for  $f$  if

$$(1 - \delta)u_f(m) + \delta u_f^f \geq (1 - \delta)Z + \delta(1 - \delta^L)\underline{u}_f^{\mathcal{R}} + \delta^{L+1}u_f^f.$$

The inequality is equivalent to

$$Z - u_f(m) \leq \delta(1 + \dots + \delta^{L-1})[u_f^f - \underline{u}_f^{\mathcal{R}}].$$

By construction,  $u_f^f - \underline{u}_f^{\mathcal{R}} > 0$ . Choose  $L$  large enough so that  $L(u_f^f - \underline{u}_f^{\mathcal{R}}) > Z - u_f(m)$ . As  $\delta \rightarrow 1$ , the LHS remains unchanged while the RHS converges to  $L(u_f^f - \underline{u}_f^{\mathcal{R}})$ , so such deviations are not profitable for  $\delta$  high enough.

*For states of the form  $\{\underline{\theta}(f, t)\}$*  Consider a one-shot deviation  $(f', W)$ . There are two cases to consider.

*Case 1:  $f' \neq f$ .* Without deviation, firm  $f'$  has payoff  $(1 - \delta^{L-t})u_{f'}(\underline{m}_f) + \delta^{L-t}u_{f'}^f$ . After deviation,  $f'$  has payoff less than  $(1 - \delta)Z + \delta(1 - \delta^L)\underline{u}_{f'}^{\mathcal{R}} + \delta^{L+1}u_{f'}^f$ . There is no profitable one-shot deviation for  $f'$  if

$$(1 - \delta^{L-t})u_{f'}(\underline{m}_f) + \delta^{L-t}u_{f'}^f \geq (1 - \delta)Z + \delta(1 - \delta^L)\underline{u}_{f'}^{\mathcal{R}} + \delta^{L+1}u_{f'}^f.$$

As  $\delta \rightarrow 1$ , the LHS converges to  $u_{f'}^f$  for all  $0 \leq t \leq L$ , while the RHS converges to  $u_{f'}^f$ . By construction,  $u_{f'}^f > \underline{u}_{f'}^{\mathcal{R}}$ . So the above inequality holds for sufficiently high  $\delta$ .

*Case 2:  $f' = f$ .* Without deviation, firm  $f'$  has payoff  $(1 - \delta^{L-t})\underline{u}_{f'}^{\mathcal{R}} + \delta^{L-t}u_{f'}^f$ . When deviating from  $\underline{m}_{f'}$ ,  $f'$  can obtain at most  $\underline{u}_{f'}^{\mathcal{R}}$ . So its payoff from deviation is at most

$$(1 - \delta)\underline{u}_{f'}^{\mathcal{R}} + \delta(1 - \delta^L)\underline{u}_{f'}^{\mathcal{R}} + \delta^{L+1}u_{f'}^f = (1 - \delta^{L+1})\underline{u}_{f'}^{\mathcal{R}} + \delta^{L+1}u_{f'}^f.$$

Firm  $f'$  has no profitable deviation if  $(1 - \delta^{L-t})\underline{u}_{f'}^{\mathcal{R}} + \delta^{L-t}u_{f'}^f \geq (1 - \delta^{L+1})\underline{u}_{f'}^{\mathcal{R}} + \delta^{L+1}u_{f'}^f$  or

$$u_{f'}^f \geq \underline{u}_{f'}^{\mathcal{R}}.$$

This is true by construction. So  $f'$  has no profitable one-shot deviation.

We have verified that there is no profitable one-shot deviation in any states of the automaton. This completes the proof.

#### REFERENCES

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