Supplemental Appendix

Self-Enforced Job Matching

Ce Liu Ziwei Wang Hanzhe Zhang

S.1 Relaxing Assumption 2

In this section, we explain how the existence result Proposition 2 can be generalized when Assumption 2 is not satisfied. As an overview, we first generalize Proposition 1 to Proposition 1*. Next, we extend Lemma 2 to Lemma 2*. Combining Lemma 2* and Proposition 1* allows us to establish the existence result without relying on Assumption 2, as presented in Proposition 2*. The proofs are presented at the end of this section.

Suppose Assumption 2 does not hold, so that there exists some firm f such that for every $\theta \in \Theta$ with $\pi(\theta) > 0$, we have $\overline{u}_f(\theta) = \underline{u}_f(\theta)$. In this case, other players' matching decisions have *no impact* on the maximum static payoff firm f can derive, since it can always turn to the unmatched workers and extract their surpluses. This means that future punishments cannot affect the matching behavior of f via dynamic incentives, nor does f find it beneficial to participate in any punishment scheme of other firms. Therefore, we can treat such a firm as "inactive" in our analysis, assign it the maximum payoff it can receive in every period, and ignore it for the rest of our analysis. An iteration is needed to identify all such firms that cannot be incentivized dynamically.

Formally, for a set of firms $\mathcal{F}' \subseteq \mathcal{F}$, denote by $Q(\mathcal{F}') \equiv \sum_{f\in\mathcal{F}'} q_f$ the total hiring capacity of $\mathcal{F}'$. When all firms in $\mathcal{F}\backslash\mathcal{F}'$ are inactive, the effective minmax payoff of a firm $f \in \mathcal{F}'$ at θ is

$$\underline{u}_f(\mathcal{F}', \theta) \equiv \min_{W'\subseteq\mathcal{W}, |W'|\leq Q(\mathcal{F}')} \ \max_{W\subseteq\mathcal{W}\backslash W', |W|\leq q_f} s(f, W, \theta).$$

Using a similar argument as in Lemma 1, we can show that this is exactly firm f's payoff from "best responding" to the worst punishment by firms in $\mathcal{F}'$, while firms in $\mathcal{F}\backslash\mathcal{F}'$ leave no surplus to their employees. Note that for each θ, the value $\underline{u}_f(\mathcal{F}', \theta)$ weakly increases as $\mathcal{F}'$ becomes smaller.

Definition 4. A hierarchical partition $\mathcal{P} \equiv \{\mathcal{P}_1, \ldots, \mathcal{P}_N, \mathcal{R}\}$ over firms $\mathcal{F}$ is induced by the following procedure. Initialize $\mathcal{P}_0 \equiv \emptyset$. For $n \geq 1$:

- If $\{f \in \mathcal{F} \backslash \bigcup_{k=0}^{n-1} \mathcal{P}_k : \overline{u}_f(\theta) = \underline{u}_f(\mathcal{F} \backslash \bigcup_{k=0}^{n-1} \mathcal{P}_k, \theta)\ \forall \theta\} \neq \emptyset$, let this set be $\mathcal{P}_n$. Assign $n = n + 1$ and continue;
- If $\{f \in \mathcal{F} \backslash \bigcup_{k=0}^{n-1} \mathcal{P}_k : \overline{u}_f(\theta) = \underline{u}_f(\mathcal{F} \backslash \bigcup_{k=0}^{n-1} \mathcal{P}_k, \theta)\ \forall \theta\} = \emptyset$, let $\mathcal{R} = \mathcal{F} \backslash \bigcup_{k=0}^{n-1} \mathcal{P}_k$ and stop.

Intuitively, each $\mathcal{P}_n$ consists of firms that cannot be punished in the matching process without cooperation from those in $\bigcup_{k=0}^{n-1} \mathcal{P}_k$. If $\mathcal{R} \neq \emptyset$, by construction, $\overline{u}_f(\theta) > \underline{u}_f(\mathcal{R}, \theta)$ for every $f \in \mathcal{R}$ and $\theta \in \Theta$. Let

$$\overline{u}_f^* \equiv \mathbb{E}_\pi[\overline{u}_f(\theta)] \quad \forall f \in \mathcal{F} \backslash \mathcal{R},$$

and

$$\underline{u}_f^*(\mathcal{R}) \equiv \mathbb{E}_\pi[\underline{u}_f(\mathcal{R}, \theta)] \quad \forall f \in \mathcal{R}.$$

A generalized version of Proposition 1 can be stated as follows.

Proposition 1*. (i) If $u \in \mathcal{U}^*$ satisfies $u_f = \overline{u}_f^*$ for all $f \in \mathcal{F} \backslash \mathcal{R}$ and $u_f > \underline{u}_f^*(\mathcal{R})$ for all $f \in \mathcal{R}$, then there is a $\underline{\delta} \in (0, 1)$ such that for every $\delta \in (\underline{\delta}, 1)$, there exists a self-enforcing matching process with firms' continuation payoffs u at the beginning of period 0. (ii) Suppose μ is a self-enforcing matching process for a given $\delta \in (0, 1)$. For every ex ante history $\overline{h} \in \overline{\mathcal{H}}$, firms' continuation payoff profile satisfies $\left(U_f(\overline{h} \,|\, \mu)\right)_{f \in \mathcal{F}} \in \mathcal{U}^*$, $U_f(\overline{h} \,|\, \mu) = \overline{u}_f^*$ for every $f \in \mathcal{F} \backslash \mathcal{R}$, and $U_f(\overline{h} \,|\, \mu) \geq \underline{u}_f^*(\mathcal{R})$ for every $f \in \mathcal{R}$.

The following is an immediate corollary of Proposition 1*. It states that if no firms can be dynamically incentivized, then a self-enforcing matching process always exists, and all such processes result in a unique profile of continuation payoffs for the firms.

Corollary 1. *Suppose $\mathcal{R} = \emptyset$. A self-enforcing matching process exists for all $0 \leq \delta < 1$; furthermore, for every self-enforcing matching process, each firm f's continuation payoff is equal to $\overline{u}_f^*$ at all ex ante histories.*

The existence of a self-enforcing matching process in this special case follows from the observation that, in the proof of Proposition 1*(i), the condition on δ is invoked only for firms in $\mathcal{R}$. For the general case of existence, we next define a random serial dictatorship with respect to the hierarchical partition $\mathcal{P} = \{\mathcal{P}_1, \ldots, \mathcal{P}_N, \mathcal{R}\}$. To do this, we first introduce a subset of orderings $\mathcal{O}_\mathcal{P} \subseteq \mathcal{O}$ that contains those that (i) give firms in $\mathcal{R}$ the highest priorities and (ii) rank firms in $\mathcal{P}_n$ higher than those in $\mathcal{P}_k$ if $n > k$. That is,

$$\mathcal{O}_\mathcal{P} \equiv \left\{ o \in \mathcal{O} : \begin{array}{c} o(\mathcal{R}) = \{1, 2, \ldots, |\mathcal{R}|\}, \text{ and} \\ \text{if } f \in \mathcal{P}_k,\ f' \in \mathcal{P}_n, \text{ and } k < n, \text{ then } o(f) > o(f') \end{array} \right\}.$$

The stage-game matching $\widehat{m}(\theta, o)$ induced by a serial dictatorship according to o is defined as in Definition 3. By using a random serial dictatorship restricted to $\mathcal{O}_\mathcal{P}$, the following lemma generalizes Lemma 2 and shows that Proposition 1*(i) is not vacuously true.

Lemma 2*. For every firm $f \in \mathcal{R}$,

$$\frac{1}{|\mathcal{O}_\mathcal{P}|} \sum_{o \in \mathcal{O}_\mathcal{P}} \mathbb{E}_\pi \Big[u_f\big(\widehat{m}(\theta, o), \theta\big)\Big] > \underline{u}_f^*(\mathcal{R}).$$

For every firm $f \in \mathcal{F} \backslash \mathcal{R}$,

$$\frac{1}{|\mathcal{O}_\mathcal{P}|} \sum_{o \in \mathcal{O}_\mathcal{P}} \mathbb{E}_\pi \Big[u_f\big(\widehat{m}(\theta, o), \theta\big)\Big] = \overline{u}_f^*.$$

In view of Lemma 2* and Proposition 1*, we can establish Proposition 2 without Assumption 2.

Proposition 2*. When firms are sufficiently patient, there exists a self-enforcing matching process in which players match according to the outcome of an RSD in every period on the path.

S.1.1 Proof of Proposition 1*

For part (i), let $(\lambda(\theta))_{\theta \in \Theta}$ be the tuple of lotteries such that $\lambda(\theta) \in \Delta(M^\circ(\theta))$ for every $\theta \in \Theta$, and $u = \mathbb{E}_\pi[\mathbb{E}_{\lambda(\theta)}[u(m, \theta)]]$. There are three cases to consider.

Case 1: $\mathcal{R} = \emptyset$. If a firm f receives $u_f = \overline{u}_f^*$ on average, it is necessary that this firm receives the highest possible payoff $\overline{u}_f(\theta)$ at every realization of θ. This in turn implies that, for each

θ, $\lambda(\theta)$ only assigns positive probability to stage-game matchings that are *stable* in a static sense. Therefore, a matching process that recommends according to $(\lambda(\theta))_{\theta\in\Theta}$ in every period is self-enforcing.

Case 2: $|\mathcal{R}| = 1$. Let f denote the single firm in $\mathcal{R}$, and let $(\underline{m}_f(\theta))_{\theta\in\Theta}$ be the matchings that give f the most severe punishment by f itself, while all other firms receive the highest possible payoff $\overline{u}_f(\theta)$. Consider the following matching process:

(I) Match according to $\lambda(\cdot)$ if $\lambda(\cdot)$ was followed in the last period or $\underline{m}_f(\cdot)$ was followed for L periods;

(II) If firm f deviates from (I), match according to $\underline{m}_f(\cdot)$ for L periods.

If firm f deviates from (II), restart (II).

It is easy to check that when L is sufficiently large, firm f has no incentive to deviate in either phase, since $\delta \to 1$. All other firms have no incentive to deviate, since they already receive maximum stage-game payoff in every period.

Case 3: $|\mathcal{R}| \geq 2$. The proof for this case essentially follows the one for Proposition 1 with proper adjustments.

For part (ii), by construction, any firm $f \in \mathcal{P}_1$ can secure a stage-game payoff $\overline{u}_f(\theta)$ by deviation at every θ regardless of the stage-game matching. Taking expectation yields $U_f(\overline{h}\,|\,\mu) = \overline{u}_f^*$ for every $f \in \mathcal{P}_1$.

Suppose $U_{f'}(\overline{h}\,|\,\mu) = \overline{u}_{f'}^*$ for every $f' \in \bigcup_{k=1}^n \mathcal{P}_k$ with $n < N$. Then all firms in $\bigcup_{k=1}^n \mathcal{P}_k$ offer zero wages to their employees. By construction, each firm $f \in \mathcal{P}_{n+1}$ can secure a stage-game payoff $\overline{u}_f(\theta)$ by deviation at every θ regardless of how firms in $\mathcal{F}\setminus\bigcup_{k=1}^n \mathcal{P}_k$ are matched in each period. Taking expectation yields $U_f(\overline{h}\,|\,\mu) = \overline{u}_f^*$ for every $f \in \mathcal{P}_{n+1}$. By induction, the equality holds for all $f \in \mathcal{F}\setminus\mathcal{R}$.

By definition of the effective minmax payoff, every firm $f \in \mathcal{R}$ can secure $\underline{u}_f(\mathcal{R}, \theta)$ in each period by deviating with workers. Taking expectation over θ yields $U_f(\overline{h}\,|\,\mu) \geq \underline{u}_f^*(\mathcal{R})$ for these firms. Rigorous proof can be adapted from that of Proposition 1(ii) in Appendix A.2.1.

S.1.2 Proof of Lemma 2*

The proof of the first statement follows that of Lemma 2.

For the second statement, take any $f \in \mathcal{P}_n$, $n = 1, 2, \ldots, N$. By definition, for every $o \in \mathcal{O}_{\mathcal{P}}$, we have $\left|W^{\#}_{o(f)}\right| < Q(\mathcal{F} \backslash \bigcup_{k=0}^{n-1} \mathcal{P}_k)$. This means

$$\begin{aligned} u_f\big(\,\widehat{m}(\theta, o), \theta\,\big) &= \max_{W \subseteq \mathcal{W} \backslash W^{\#}_{o(f)}, |W| \leq q_f} s(f, W, \theta) \\ &\geq \min_{W' \subseteq \mathcal{W}, |W'| \leq Q(\mathcal{F} \backslash \bigcup_{k=0}^{n-1} \mathcal{P}_k)} \max_{W \subseteq \mathcal{W} \backslash W', |W| \leq q_f} s(f, W, \theta) \\ &= \overline{u}_f(\theta), \quad \forall \theta \in \Theta, \end{aligned}$$

where the last equality comes from the definition of $\mathcal{P}_n$. Taking expectation over Θ gives $\mathbb{E}_\pi\big[u_f\big(\,\widehat{m}(\theta, o), \theta\,\big)\big] = \overline{u}^*_f$, which suffices for the second statement to hold.

S.2 Multi-Firm Deviation Plans

We now generalize our model to study deviation plans that involve more than one firm. As discussed in the main text, we assume that whenever a deviation from the process happens (i.e., the realized stage-game matching differs from the default specified by the matching process in some period), the set of firms in the deviation plan can be identified and recorded. Formally, a t-period ex ante history is defined as $\overline{h} = (\theta_\tau, \gamma_\tau, m_\tau, F_\tau)_{\tau=0}^{t-1}$, where $F_\tau \subseteq \mathcal{F}$ records all the firms in the deviation plan (if any) responsible for the blocking in period τ, while an empty set $\emptyset$ indicates the realized stage-game matching m_τ follows the matching process being studied. As before, $\overline{\mathcal{H}}_t$ denotes the set of all t-period ex ante histories, $\overline{\mathcal{H}} \equiv \bigcup_{t=0}^{\infty} \overline{\mathcal{H}}_t$ the set of all ex ante histories, $\mathcal{H}_t \equiv \overline{\mathcal{H}}_t \times \Theta \times \Gamma$ the set of t-period ex post histories, and $\mathcal{H} \equiv \overline{\mathcal{H}} \times \Theta \times \Gamma$ the set of all ex post histories. A matching process is then $\mu : \mathcal{H} \to M$.

Recall that a deviation plan by a single firm f is a pair $(d : \mathcal{H} \to 2^{\mathcal{W}}, \eta : \mathcal{H} \to \mathbb{R}^{|\mathcal{W}|})$ such that $|d(h)| \leq q_f$ for any h and $\eta_w(h) \neq 0$ only if $w \in d(h)$. For a set of firms $F \in 2^{\mathcal{F}} \backslash \{\emptyset\}$, a joint deviation plan $(F, \{(d_f, \eta_f)\}_{f \in F})$ by firms F is a collection of deviation plans $\{(d_f, \eta_f)\}_{f \in F}$, such that (i) $d_f(h) \cap d_{f'}(h) = \emptyset$ for all $f \neq f'$, $h \in \mathcal{H}$ and (ii) for each $f \in F$, $(d_f(h), \eta_f(h))$ differs from the pair specified by $\mu(h)$ at some $h \in \mathcal{H}$. The following is the multiple-firm counterpart of Assumption 1.

Assumption 3. *Let* $[m, (F, \{(\widehat{W}_f, \widehat{p}_f)\}_{f \in F})] \in M$ *denote the stage-game matching that is* *realized after coalitional deviation* $(F,\{(\widehat{W}_f,\widehat{p}_f)\}_{f\in F})$ *from stage-game matching* $m=(\phi,p)$, *and let* (ϕ',p') *denote the assignment and wages in* $[m,(F,\{(\widehat{W}_f,\widehat{p}_f)\}_{f\in F})]$. *We assume that the assignment* ϕ' *satisfies* $\phi'(f)=\widehat{W}_f$ *for all* $f\in F$ *and* $\phi'(f')=\phi(f')\setminus\bigcup_{f\in F}\widehat{W}_f$ *for every* $f'\notin F$*; furthermore, the wages satisfy* $p'_f=\widehat{p}_f$ *for all* $f\in F$*, while* $p'_{f'w}=p_{f'w}$ *for every* $f'\notin F$ *and* $w\in\phi'(f')$.

Given a matching process μ and a joint deviation plan $(F,\{(d_f,\eta_f)\}_{f\in F})$, the manipulated matching process, denoted by $[\mu,(F,\{(d_f,\eta_f)\}_{f\in F})]:\mathcal{H}\to M$, is a matching process defined by

$$\Big[\mu,(F,\{(d_f,\eta_f)\}_{f\in F})\Big](h)\equiv\Big[\mu(h),(F,\{(d_f(h),\eta_f(h))\}_{f\in F})\Big]\quad\forall h\in\mathcal{H}.$$

The joint deviation plan $(F,\{(d_f,\eta_f)\}_{f\in F})$ from μ is *feasible* if for every $f\in F$, at every ex post history $h=\big(\overline{h},\theta,\gamma,F\big)$ such that $d_f(h)\neq\mu(f|h)$,

$$v_w\Big(\Big[\mu,(F,\{(d_f,\eta_f)\}_{f\in F})\Big](h),\,\theta\Big)>v_w\Big(\mu(h),\theta\Big)\quad\forall w\in d_f(h).\tag{20}$$

Remark. Observe that any worker poached in a multi-firm deviation belongs to the blocking coalition and therefore must be better off than following the recommended stage-game matching. Therefore, if a firm f can poach workers W when jointly deviating with other firms from a stage-game matching m, it can use the same wage offers to poach those workers when it is the only firm that is deviating from m.

Finally, the joint deviation plan $(F,\{(d_f,\eta_f)\}_{f\in F})$ is *profitable* if there exists an ex post history h such that $U_f\big(h\,\big|\,[\mu,(F,\{(d_f,\eta_f)\}_{f\in F})]\big)>U_f(h|\mu)$ for *all* $f\in F$.

A matching process μ is *strongly self-enforcing* if (i) $v_w(\mu(h),\theta)\geq 0$ for every $w\in\mathcal{W}$ at every ex post history $h\in\mathcal{H}$ and (ii) there does not exist a nonempty set of firms that has a feasible and profitable joint deviation plan.

Proposition 2′. When firms are sufficiently patient, there exists a strongly self-enforcing matching process in which players match according to the outcome of an RSD in every period on path.

Proof. For simplicity, we prove the result under Assumption 2, but it can be easily generalized using the arguments in Supplemental Appendix S.1. Fix an indexing of the finite set of firms $\mathcal{F}$ by the numbers $\{1, 2, \ldots, |\mathcal{F}|\}$. Pick $u \in \mathcal{U}^*$. The tuples $(\lambda(\theta))_{\theta\in\Theta}$ and $(\lambda^f(\theta))_{\theta\in\Theta}$ are defined as in the proof of Proposition 1(i). Recall that $\{\underline{m}_f(\theta)\}_{\theta\in\Theta, f\in\mathcal{F}}$ are the minmax stage-game matchings constructed in Lemma 5. Consider a matching process μ with the following three phases:

(I) If past realizations from $\lambda(\cdot)$ were followed: Match according to $\lambda(\cdot)$;

(II) If $\lambda(\cdot)$ was not followed and F is the set of firms responsible for the coalitional deviation, let f be the firm with the lowest index in F: For the next L periods, match according to $\underline{m}_f(\cdot)$.

(III) If $\underline{m}_f(\cdot)$ was followed for L periods: Match according to the realization from $\lambda^f(\cdot)$ until a firm deviates.

If there is a deviation from (II) or (III) and F' is the set of firms responsible for the coalitional deviation, restart (II) with f replaced by f', the firm with the lowest index in F'.

Lemma 8. *If there exists a feasible and profitable joint deviation plan $(F, \{(d_f, \eta_f)\}_{f\in F})$ from μ, then there exists a feasible and profitable deviation plan $(\widehat{d}, \widehat{\eta})$ of a single firm $\widehat{f}$ when it is sufficiently patient.*

Proof. Denote by $\widehat{h} = (\bar{h}, \theta, \gamma)$ the ex post history such that $[\mu, F, \{(d_f, \eta_f)\}_{f\in F}](\widehat{h}) \neq \mu(\widehat{h})$ and $U_f\big(\widehat{h}\,\big|\,[\mu, (F, \{(d_f, \eta_f)\}_{f\in F})]\big) - U_f(\widehat{h}|\mu) > 0$ for all $f \in F$; when there are multiple such histories, pick an arbitrary one. Suppose $\widehat{h}$ is a $\widehat{t}$-period ex post history. Let $\widehat{f}$ be the firm with the lowest index in F. Define a deviation plan $(\widehat{d}, \widehat{\eta})$ for firm $\widehat{f}$ as follows:

- At $\widehat{h}$, make a feasible deviation that leads to $\widehat{m} \equiv [\mu(\widehat{h}), (\widehat{f}, \widehat{W}, \widehat{p})] \neq \mu(\widehat{h})$ and triggers phase (II). Note that this can always be achieved by hiring a different group of workers with sufficiently high wages.

- Let $h_F = (\bar{h}, (\theta_\tau, \gamma_\tau, m_\tau, F_\tau)_{\tau=\widehat{t}}^{t-1}, (\theta_t, \gamma_t))$ be an ex post history with $t > \widehat{t}$ that satisfies:

    1. h_F is generated by following the manipulated process $[\mu, F, \{(d_f, \eta_f)\}_{f\in F}]$ from $\widehat{h}$ given the realizations $(\theta_\tau, \gamma_\tau)_{\tau=\widehat{t}}^{t-1}$;

2. $\mu(h_F)$ is either in phase (II) $\big($i.e., specifying $\underline{m}_{\widehat{f}}(\theta_t)\big)$ or in phase (III) $\big($i.e., randomizing according to $\lambda^{\widehat{f}}(\theta_t)\big)$;

3. $\widehat{f}$ actively carries out a deviation at h_F under the joint deviation plan $(F, \{(d_f, \eta_f)\}_{f\in F})$ $\big($i.e., $(d_{\widehat{f}}(h_F), \eta_{\widehat{f}}(h_F)) \neq \mu(\widehat{f}|h_F)\big)$.

For each t-period ex post history h following $\widehat{h}$ such that (i) the realizations $(\theta_\tau, \gamma_\tau)_{\tau=\widehat{t}}^{t-1}$ are the same in h and h_F, and (ii) $\mu(h)$ is in the same phase as $\mu(h_F)$, let $(\widehat{d}(h), \widehat{\eta}(h)) = (d_{\widehat{f}}(h_F), \eta_{\widehat{f}}(h_F))$.

- For any other ex post history h, let $(\widehat{d}(h), \widehat{\eta}(h)) = \mu(\widehat{f}|h)$.

In words, in the constructed single-firm deviation plan, firm $\widehat{f}$ mimics its own deviating behavior in the joint deviation plan only when the manipulated process is either in the punishment phase (II) or in the reward phase (III).

By construction, for every ex post history h such that $(\widehat{d}(h), \widehat{\eta}(h)) \neq \mu(\widehat{f}|h)$, there exists some h_F such that $\mu(h) = \mu(h_F)$ and $(\widehat{d}(h), \widehat{\eta}(h)) = (d_{\widehat{f}}(h_F), \eta_{\widehat{f}}(h_F))$. Since $(F, \{(d_f, \eta_f)\}_{f\in F})$ is feasible, we have

$$\begin{aligned}
v_w\Big(\Big[\mu, (\widehat{f}, \widehat{d}, \widehat{\eta})\Big](h), \, \theta\Big) &= v_w\Big(\Big[\mu(h_F), (\widehat{f}, d_{\widehat{f}}(h_F), \eta_{\widehat{f}}(h_F))\Big], \, \theta\Big) \\
&= v_w\Big(\Big[\mu, (F, \{d_f, \eta_f\}_{f\in F})\Big](h_F), \, \theta\Big) \\
&> v_w\Big(\mu(h_F), \theta\Big) \\
&= v_w\Big(\mu(h), \theta\Big) \qquad \forall w \in d_f(h).
\end{aligned}$$

Therefore, the deviation plan $(\widehat{d}, \widehat{\eta})$ of firm $\widehat{f}$ is feasible.

The processes $[\mu, (\widehat{f}, \widehat{d}, \widehat{\eta})]$ and $[\mu, (F, \{(d_f, \eta_f)\}_{f\in F})]$ induce two measures over the set of outcomes $\overline{\mathcal{H}}_\infty = (\Theta \times \Gamma \times M)^\infty$, where we suppress the record of deviating firms $F \in 2^{\mathcal{F}}$ as it does not influence continuation payoffs. Note that conditional on each realized sequence of the state and the public randomization device $(\theta_\tau, \gamma_\tau)_{\tau=0}^t$, any process must induce a point mass on some stage-game matching in period t. For $t > \widehat{t}$, fixing the ex post history $\widehat{h}$ and any sequence $(\theta_\tau, \gamma_\tau)_{\tau=\widehat{t}+1}^t$, if firm $\widehat{f}$ is matched to different sets of workers or pays different wage vectors under two manipulated processes, firm $\widehat{f}$ matches according to $\lambda^{\widehat{f}}(\theta_t)$ and the realized γ_t under $[\mu, (\widehat{f}, \widehat{d}, \widehat{\eta})]$, while $\widehat{f}$ matches according to $\underline{m}_{\widehat{f}}(\theta_t)$ or deviates from it under $[\mu, (F, \{(d_f, \eta_f)\}_{f\in F})]$. By the fact that $u^{\widehat{f}}_{\widehat{f}} > \underline{u}^*_{\widehat{f}}$ and parts (i) and (ii) of Lemma 5, we can conclude that in every period $t > \widehat{t}$, firm $\widehat{f}$'s expected payoff averaged over θ_t and γ_t under $[\mu, (\widehat{f}, \widehat{d}, \widehat{\eta})]$ is weakly higher than that under $[\mu, (F, \{(d_f, \eta_f)\}_{f\in F})]$. Thus, we have

$$\begin{aligned} U_{\widehat{f}}\big(\widehat{h} \,\big|\, [\mu, (\widehat{f}, \widehat{d}, \widehat{\eta})]\big) - U_{\widehat{f}}(\widehat{h}|\mu) &= \Big[U_{\widehat{f}}\big(\widehat{h} \,\big|\, [\mu, (\widehat{f}, \widehat{d}, \widehat{\eta})]\big) - U_{\widehat{f}}\big(\widehat{h} \,\big|\, [\mu, (F, \{(d_f, \eta_f)\}_{f\in F})]\big)\Big] \\ &\quad + \Big[U_{\widehat{f}}\big(\widehat{h} \,\big|\, [\mu, (F, \{(d_f, \eta_f)\}_{f\in F})]\big) - U_{\widehat{f}}(\widehat{h}|\mu)\Big] \\ &\geq (1-\delta)\Big(u_f(\widehat{m}, \theta_{\widehat{t}}) - u_f\big([\mu, (F, \{d_f, \eta_f\}_{f\in F})](\widehat{h}), \theta_{\widehat{t}}\big)\Big) \\ &\quad + \Big[U_{\widehat{f}}\big(\widehat{h} \,\big|\, [\mu, (F, \{(d_f, \eta_f)\}_{f\in F})]\big) - U_{\widehat{f}}(\widehat{h}|\mu)\Big]. \end{aligned}$$

Since $U_{\widehat{f}}\big(\widehat{h} \,\big|\, [\mu, (F, \{(d_f, \eta_f)\}_{f\in F})]\big) - U_{\widehat{f}}(\widehat{h}|\mu) > 0$ due to the profitability of $(F, \{(d_f, \eta_f)\}_{f\in F})$, when δ is sufficiently close to 1, we have $U_{\widehat{f}}\big(\widehat{h} \,\big|\, [\mu, (\widehat{f}, \widehat{d}, \widehat{\eta})]\big) - U_{\widehat{f}}(\widehat{h}|\mu) > 0$, which means the deviation plan $(\widehat{d}, \widehat{\eta})$ is also profitable for firm $\widehat{f}$. □

Intuitively, under μ, whenever a joint deviation plan $(F, \{(d_f, \eta_f)\}_{f\in F})$ is formed, the firm with the lowest index in F is singled out as the "scapegoat" of the coalition. This firm is punished as if it were the sole deviator, and the punishment scheme restarts if any member of the group fails to follow the matching process. This construction implies that if the scapegoat were to unilaterally implement a deviation plan that replicates its own behavior in the joint deviation, it would be punished less frequently under the manipulated process. As a result, whenever a feasible and profitable joint deviation plan exists, there also exists an alternative feasible and profitable single-firm deviation plan that yields a continuation payoff arbitrarily close to that in the joint deviation, as $\delta \to 1$.

Hence, by Lemma 3, it suffices to rule out feasible and profitable *one-shot* deviations by a single firm. Since the matching process μ constructed above reduces to the one in the proof of Proposition 1 when restricted to deviations by a single firm, and we have shown that no feasible and profitable one-shot deviation exists under such a process, this completes the proof. □